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Monte Carlo simulations of the NJL model near the nonzero temperature phase transition

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Abstract: We present results from numerical simulations of the Nambu−Jona-Lasinio model with an $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ chiral symmetry and $N_c = 4, 8$, and 16 quark colors at nonzero temperature. We performed the simulations by utilizing the hybrid Monte Carlo and hybrid Molecular Dynamics algorithms. We show that the model undergoes a second order phase transition. The critical exponents measured are consistent with the classical $3d$ O(4) universality class and hence in accordance with the dimensional reduction scenario. We also show that the Ginzburg region is suppressed by a factor of $1/N_c$ in accordance with previous analytical predictions.

Keywords: Lattice Quantum Field Theory, Thermal Field Theory.

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Contents

1. Introduction

The behavior of symmetries at finite temperature is one of the most outstanding problems in major areas of particle physics such as cosmology, relativistic heavy-ion collisions, and the quark-gluon plasma. In recent years, considerable work has been done on the physics of the finite temperature chiral phase transition in QCD and related models. Since the problem of chiral symmetry breaking and its restoration is intrinsically non-perturbative, the number of techniques available is limited. Lattice simulations have so far provided us with most of our knowledge about this phenomenon. Various lattice results indicate that the QCD transition is second order [1]. However, due to problems with finite size effects near the critical temperature and the proper inclusion of fermions on the lattice, these calculations are still not fully satisfactory. Therefore, an important role is played by model field theories which incorporate certain basic features of QCD and they are easier to deal with. Such a theory is the Nambu−Jona-Lasinio (NJL) model. This model was introduced in the sixties as a theory of interacting nucleons [3] and later it was reformulated in terms of quark degrees of freedom. In this paper we present results from numerical simulations of the NJL model near the thermal phase transition.

A compelling idea based on universality and dimensional reduction [2], is that in QCD with two massless quarks the physics near the transition can be described by the three-dimensional O(4)-symmetric sigma model. The reasoning behind the dimensional reduction scenario is based on the dominance of the light degrees of freedom near the critical temperature T_c. In the imaginary time formalism of finite temperature field theory, a given system is defined on a manifold $S^1 \times \mathbb{R}^d$, where d is the number of spatial dimensions. The inverse temperature is the circumference of S^1. The boundary condition in the time direction is periodic for bosons and anti-periodic for fermions. At temperatures larger than the physical scales of the system, the nonzero Matsubara modes acquire a heavy mass. Therefore,

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the non-static configurations are strongly suppressed in the Boltzmann sum. If the original theory has fermions, all fermionic modes become massive due to the anti-periodicity in the temporal direction and can be integrated over. This is not to say that non-static modes have no effect. Rather, they generate counter-terms to the three-dimensional theory. It is not yet clear whether the dimensional reduction is the correct description of the QCD phase transition.

In the continuum, the NJL model is described by the euclidean lagrangian density

$$\mathcal{L} = \overline{\psi}_i \partial\!\!\!/ \psi_i - \frac{g^2}{2N_c} \left[\left(\overline{\psi}_i \psi_i \right)^2 - \left(\overline{\psi}_i \gamma_5 \vec{\tau} \psi_i \right)^2 \right], \tag{1.1}$$

where ψ ($\bar{\psi}$) are the quark (anti-quark) fields and $\vec{\tau} \equiv (\tau_1, \tau_2, \tau_3)$ are the Pauli spin matrices, which run over the internal $\mathrm{SU}(2)_I$ isospin symmetry. The index i runs over N_c quark colors and g^2 is the coupling constant of the four-fermion interaction. The model is chirally symmetric under $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$: $\psi \to (P_L U + P_R V)\psi$, where U and V are independent global SU(2) rotations and the operators $P_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5)$ project onto left and right handed spinors, respectively. In other words the explicit flavor symmetry gives rise to $N_f = 2$. It is also invariant under $\mathrm{U}(1)_V$ corresponding to a conserved baryon number. The theory becomes easier to treat, both analytically and numerically, if we introduce scalar and pseudo-scalar auxiliary fields denoted by σ and $\vec{\pi}$, respectively. The bosonised lagrangian is

$$\mathcal{L} = \overline{\psi}_i \left(\partial\!\!\!/ + \sigma + i\gamma_5 \vec{\pi} \cdot \vec{\tau} \right) \psi_i + \frac{N_c}{2g^2} \left(\sigma^2 + \vec{\pi} \cdot \vec{\pi} \right). \tag{1.2}$$

Since the theory has no gauge fields it fails to produce the physics of confinement. The dynamic generation of fermion masses, though, brought about by the breaking of chiral symmetry to $\mathrm{SU}(2)_I$ at g^2 larger than a critical g_c^2 is one of the important properties of the model. This implies the creation of both the Nambu-Goldstone bosons (π) and the chiral partner sigma (σ), which are quark-antiquark bound states. The interaction strength has a mass dimension -2, implying that the model is non-renormalizable. The triviality of the NJL model was also demostrated numerically in [4]. This model works well in the intermediate scale region as an effective theory of QCD with a momentum cutoff comparable to the scale of chiral symmetry breaking $\Lambda_{\chi SB} \approx 1$GeV. Various authors used the NJL model to study properties of hadrons (for a review see [5]). It was also shown that the model has an interesting phase diagram in the (T, μ) plane, where μ is the quark chemical potential. This phase diagram is in close agreement with predictions from other effective field theories of QCD. In the Hartree-Fock approximation, the theory has a tricritical point in the (T, μ) plane [6] when the quark current mass, the coupling g, and the momentum cut-off Λ are chosen to have physically meaningful values. Recently, lattice simulations showed that the ground-state at high μ and low T is that of a traditional BCS superfluid [7].

A possible loophole of the dimensional reduction scenario is that in the NJL model (and in QCD) the meson states are composite quark−anti-quark bound states whose density and size increase as $T \to T_c$. This may imply that the fermionic sub-structure is apparent on physical length scales and as a consequence we may have a maximal violation

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of the bosonic character of mesons near T_c. If this is indeed the case, then the quarks become essential degrees of freedom irrespective of how heavy they are. Several authors observed [8, 9] that in the large N_c limit, where quarks are the only degrees of freedom, the exponents of the finite temperature transition are given by the Landau-Ginzburg Mean Field (MF) theory. The reasoning behind this result is that at finite temperature the zero fermionic modes are absent and therefore, contrary to a purely bosonic theory, the effect of making the temporal direction finite ($1/T$) is to regulate the infrared behavior and suppress fluctuations. However, inclusion of mesonic fluctuations in a non-perturbative scheme that takes into account $1/N_c$ effects results in a first order finite temperature phase transition [10]. The first order transition discontinuity decreases with decreasing the cut-off that controls the strength of the mesonic fluctuations, but the nature of the transition remains unchanged even for very small values of the cut-off. It was also shown numerically in both the $d = (2 + 1)$ [11, 12] and $d = (3 + 1)$ [13] cases that the Z_2-symmetric model undergoes a second order finite temperature phase transition. This transition belongs to the $(d - 1)$-dimensional Ising model universality, implying that the dimensional reduction scenario is a valid description. In this paper, we study the case where the chiral symmetry is continuum.

In section 2 we summarize the lattice formulation of the model. In section 3 we present the observables we used in the analysis of the critical properties of the theory. Finally, in section 4 we present analysis of data in the vicinity of the critical temperature and in the low temperature broken phase. We show that the model undergoes a second order phase transition, which belongs to the $3d$ O(4) universality class, and that the nontrivial scaling region is suppressed by a factor of $1/N_c$ in accordance with results presented in [11].

2. Lattice Model

The lattice action of the NJL model used in this study was the one first used in [4],

$$S = \sum_{\alpha=1}^{N}\sum_{xy}\Big(\bar{\chi}_\alpha(x)M[\sigma,\vec{\pi}]_{xy}\chi_\alpha(y) + \bar{\zeta}_\alpha(x)M^\dagger[\sigma,\vec{\pi}]_{xy}\zeta_\alpha(y)\Big) + \\ + \frac{2N}{g^2}\sum_{\tilde{x}}(\sigma^2(\tilde{x}) + \vec{\pi}(\tilde{x})\cdot\vec{\pi}(\tilde{x}))\,, \tag{2.1}$$

where χ, $\bar{\chi}$, ζ, and $\bar{\zeta}$ are Grassmann-valued staggered fermion fields defined on the sites x of a $(3+1)d$ euclidean lattice. The scalar field σ and the pseudoscalar triplet $\vec{\pi}$ are defined on the dual lattice $\tilde{x}$ and the index α runs over N staggered fermion species. The kinetic operator M is the usual one for four-fermion models with staggered fermions, modified to incorporate the SU(2) $\otimes$ SU(2) chiral symmetry and is given by

$$M_{xy} = \left(\frac{1}{2}\sum_{\mu}\eta_\mu(x)[\delta_{y,x+\hat{\mu}} - \delta_{y,x-\hat{\mu}}] + m\delta_{xy}\right)\delta_{\alpha\beta}\delta_{pq} + \\ + \frac{1}{16}\delta_{xy}\delta_{\alpha\beta}\sum_{\langle\tilde{x},x\rangle}(\sigma(\tilde{x}) + i\varepsilon(x)\vec{\pi}(\tilde{x})\cdot\vec{\tau}_{pq})\,, \tag{2.2}$$

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where m is the bare fermion mass and the $\mathrm{SU}(2)_I$ indices p, q are shown explicitly. The phase factors $\epsilon(x) \equiv (-1)^{(x_0+x_1+x_2+x_3)}$ and $\eta_\nu = (-1)^{x_0+\ldots+x_{\nu-1}}$ are the remnants of the γ_5 and γ_ν Dirac matrices on the lattice. The symbol $\langle \tilde{x}, x \rangle$ denotes the set of 16 dual sites $\tilde{x}$ adjacent to the direct lattice site x.

It can be easily shown that in the limit $m \to 0$, the action given by eq. (2.1) has a global $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ [4]. In the continuum limit, fermion doubling leads to a physical content for the model of $8N$ fermion species - $4N$ described by χ and $4N$ by ζ. The extra factor of 2 over the usual relation for four-dimensional gauge theories arises from the impossibility of even/odd partitioning in the dual-site approach to lattice four-fermi models. We refer to these degrees of freedom as colors and hence define $N_c \equiv 8N$. Further details about the lattice action are given in [4].

We performed lattice simulations with $N_c = 4, 8$, and 16. The $N_c = 4$ simulation necessitates a fractional $N = 0.5$. This can be achieved using the hybrid Molecular Dynamics (HMD) algorithm. More specifically, we used the "R-algorithm" of Gottlieb et al. [14], for which the systematic error is $\mathrm{O}(N_c^2 dt^2)$ [4], where dt is the discrete simulation time-step. We checked that these systematic errors are smaller than the statistical errors of the various observables used to study the critical properties of the model. On the other hand, the $N_c = 8$ and 16 simulations were performed by employing the exact hybrid Monte Carlo (HMC) algorithm. The lengths of the hamiltonian trajectories between momentum refreshments were drawn from a Poisson distribution with mean 2.0. For the inversion of matrix M of eq. (2.2), we used the conjugate gradient method.

The simulations were performed on asymmetric lattices with L_s lattice spacings a in the three spatial dimensions, $L_t \ll L_s$ lattice spacings in the temporal direction, and volume $V \equiv L_s^3 \cdot L_t$. The spatial extent must be much larger than the correlation length of the order parameter in order to obtain results that are close to the thermodynamic limit. The temperature T is given by $1/(L_t a)$. The temperature of the system may be altered either by varying $\beta \equiv 1/g^2$, which amounts to varying the lattice spacing or by varying L_t. In our simulations, we kept L_t fixed and varied the lattice spacing.

3. Observables

In order to study the chiral symmetry of the model we work in the chiral limit. Hence, we choose not to introduce a bare quark mass into the lattice action. Without the benefit of this interaction, the direction of symmetry breaking changes over the course of the simulation such that $\Sigma \equiv \frac{1}{V}\sum_x \sigma(x)$ and $\Pi_i \equiv \frac{1}{V}\sum_x \pi_i(x)$ average to zero over the ensemble. It is in this way that the absence of spontaneous symmetry breaking on a finite lattice is enforced. Another option is to introduce an effective order parameter Φ equal to the magnitude of the vector $\vec{\Phi} \equiv (\Sigma, \vec{\Pi})$. In the thermodynamic limit $\langle \Phi \rangle$ is equal to the true order parameter $\langle \sigma \rangle$ extrapolated to zero quark mass.

The Finite Size Scaling (FSS) method [15] is a well-established tool for studying phase transitions. We employ this method to study the critical behavior of the model on lattices available to us. The correlation length ξ on a finite lattice is limited by the size of the

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system and consequently no true criticality can be observed. The dependence of a given thermodynamic observable, A, on the size L_s of the box is singular. According to the FSS hypothesis, in the large volume limit, A is given by:

$$A(t, L_s) = L_s^{\rho_A/\nu} Q_A(t L_s^{1/\nu}) \,, \tag{3.1}$$

where $t \equiv (\beta_c - \beta)/\beta_c$ is the reduced temperature, ν is the exponent of the correlation length, Q_A is a scaling function that is not singular at zero argument, and ρ_A is the critical exponent for the quantity A. Using eq. (3.1), one can determine such exponents by measuring A for different values of L_s.

In the large L_s limit, the FSS scaling form for the thermal average of the effective order parameter $\langle \Phi \rangle$ is given by

$$\langle \Phi \rangle = L_s^{-\beta_{\rm mag}/\nu} f_\sigma(t L_s^{1/\nu}) \,. \tag{3.2}$$

Another quantity of interest is the susceptibility χ that is given, in the static limit of the fluctuation-dissipation theorem, by

$$\chi = \lim_{L_s \to \infty} V[\langle \vec{\Phi}^2 \rangle - \langle \vec{\Phi} \rangle \cdot \langle \vec{\Phi} \rangle] \,, \tag{3.3}$$

where V is the lattice volume. For finite systems, the true order parameter $\langle \vec{\Phi} \rangle$ vanishes and for $\beta \geq \beta_c$ the susceptibility is given by:

$$\chi = V \langle \Phi^2 \rangle \,. \tag{3.4}$$

This relation should scale at criticality like $L_s^{\gamma/\nu}$. For small L_s, corrections to the FSS relations become important.

Furthermore, logarithmic derivatives of $\langle \Phi^j \rangle$ can give additional estimates for ν. It is easy to show that

$$D_j \equiv \frac{\partial}{\partial \beta} \ln \langle \Phi^j \rangle = \left[\frac{\langle \Phi^j S_b \rangle}{\langle \Phi^j \rangle} - \langle S_b \rangle \right] , \tag{3.5}$$

where $S_b \equiv 2N \sum_{(\tilde{x})} (\sigma^2(\tilde{x}) + \vec{\pi}(\tilde{x}) \cdot \vec{\pi}(\tilde{x}))$ is the purely bosonic part of the lattice action (eq. (2.1)). D_j has a scaling relation

$$D_j = L_s^{1/\nu} f_{D_j}(t L_s^{1/\nu}) \,. \tag{3.6}$$

4. Results

In this section we present the analysis of numerical results in both the vicinity of the finite temperature phase transition and the low temperature chirally broken phase. The HMD R algorithm was used to simulate the NJL model with $N_c = 4$ whereas the HMC algorithm was used for $N_c = 8$ and 16.

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4.1 Finite size scaling

In this section we present the FSS analysis for the $N_c = 8$ NJL model in the vicinity of the chiral phase transition. The lattice sizes ranged from $L_s = 16$ to $L_s = 54$ with fixed $L_t = 4$. We used the histogram reweighting method [16] to perform our study most effectively. This enabled us to calculate the observables in a region of couplings around the simulation coupling. We utilized the reweighting technique efficiently by performing simulations at slightly different couplings β_i close to the critical coupling β_c. Details of the simulations are listed in table 1. The expectation values of various observables $A_{L_s}^{(\beta_i)}(\beta)$ and the associated statistical errors ΔA_i were obtained by jackknife blocking. They were then combined in a single expression $A \equiv A_{L_s}(\beta)$ to obtain, as in [17],

$$A = \left[\frac{A_1}{\Delta A_1} + \cdots + \frac{A_i}{\Delta A_i}\right](\Delta A)^2\,, \tag{4.1}$$

where ΔA is given by

$$\frac{1}{(\Delta A)^2} = \frac{1}{(\Delta A_1)^2} + \cdots + \frac{1}{(\Delta A_i)^2}\,. \tag{4.2}$$

In figure 1 we show the results of this procedure for the susceptibility χ defined in eq. (3.4).

Near β_c the expansion of the FSS scaling relation for χ, as in [18], is:

$$\begin{aligned} \chi &= a(t) + b(x)L_s^{\gamma/\nu}\,, \\ a(t) &\equiv a_0 + a_1 t + a_2 t^2 + \cdots \\ b(x) &\equiv b_0 + b_1 x + b_2 x^2 + \cdots \qquad \text{where} \qquad x \equiv tL_s^{1/\nu}\,. \end{aligned} \tag{4.3}$$

For large enough L_s one can neglect $a(t)$. We fitted the data generated on lattices with $L_s = 16, \ldots, 54$ to eq. (4.3) by including up to the linear terms in the expansions of $a(t)$ and $b(t)$. The data used in the fit are shown in figure 1. We extracted $\beta_c = 0.52589(9)$, $\nu = 0.747(20)$ and $\gamma/\nu = 1.94(5)$ with $\chi^2/DOF = 0.5$. The values of the exponents are in good agreement with the $3d$ O(4) universality exponents $\nu = 0.7479(90)$ and $\gamma/\nu = 1.97(1)$ reported in [19]. The possibility of a first order phase transition is clearly excluded, because

L_s	β_i	Trajectories
16	0.5250	220,000
16	0.5260	440,000
22	0.5250	200,000
22	0.5260	290,000
30	0.5250	100,000
30	0.5260	160,000
40	0.5250	65,000
40	0.5260	66,000
54	0.5255	40,000
54	0.5260	50,000

Table 1: Simulations for the FSS analysis.

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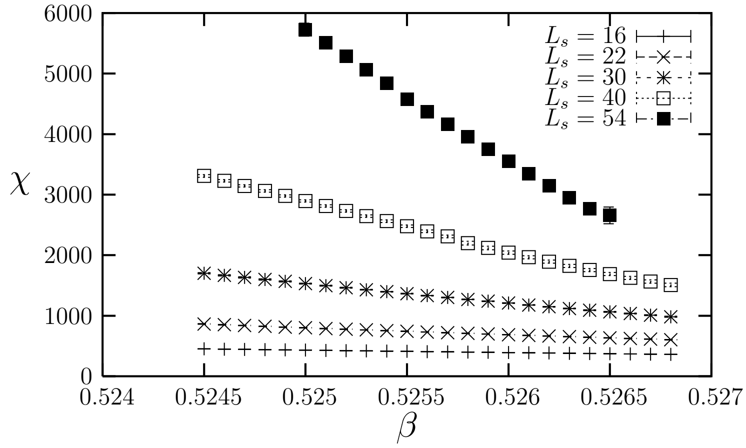

Figure 1: The susceptibility χ vs. β for different lattice volumes.

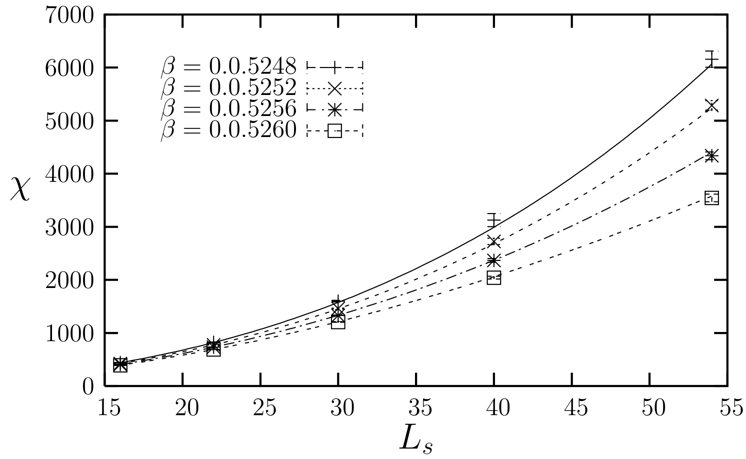

Figure 2: The susceptibility χ vs. L_s for different values of β.

in such a scenario $\gamma/\nu = 3$ (the number of the spatial dimensions). The high quality of this "global" fit is serious evidence that the reported value for the critical coupling is accurate. If we neglect the $a(t)$ terms then in order to get $\beta_c = 0.52572(16)$, $\nu = 0.747(39)$ that are consistent with the results reported above we also have to discard the $L_s = 16, 22$ data. Inclusion of the smaller lattices gives results for ν and γ/ν that are significantly smaller than the expected $3d$ O(4) exponents. Our analysis implies that finite size effects are relatively large and therefore one has to use large lattices in order to reach the asymptotic FSS regime. In figure 2 we plot χ versus L_s for four different values of β.

Next, we present the FSS analysis of the logarithmic derivatives D_j defined in eq. (3.5). We fitted the $L_s \geq 22$ data to the linear expansion of the FSS relation of D_j,

$$D_j(\beta, L_s) \simeq c_{1j} L_s^{1/\nu} + c_{2j}(\beta_c - \beta) L_s^{2/\nu} \,, \tag{4.4}$$

with fixed $\beta_c = 0.52589$ (obtained by the analysis of the susceptibility). The results displayed in table 2 show that the values of ν are in good agreement with the $3d$ O(4) result $\nu = 0.7479(90)$ [19]. We should also note that if we include the $L_s = 16$ data in the

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D_j	ν	χ^2/DOF
	$L_s = 22 - 54$	
D_1	0.750(8)	0.7
D_2	0.749(6)	0.9
D_3	0.747(6)	0.8
	$L_s = 16 - 54$	
D_1	0.707(4)	1.6
D_2	0.714(3)	1.7
D_3	0.713(3)	1.6

Table 2: Results from fits to eq. (4.4) for different ranges of L_s.

D_j	ν	χ^2/DOF
D_1	0.721(23)	0.08
D_2	0.721(21)	0.15
D_3	0.722(20)	0.31

Table 3: Results from fits to $D_j \sim L_s^{1/\nu}$ for $22 \leq L_s \leq 54$ at $\beta = 0.52589$.

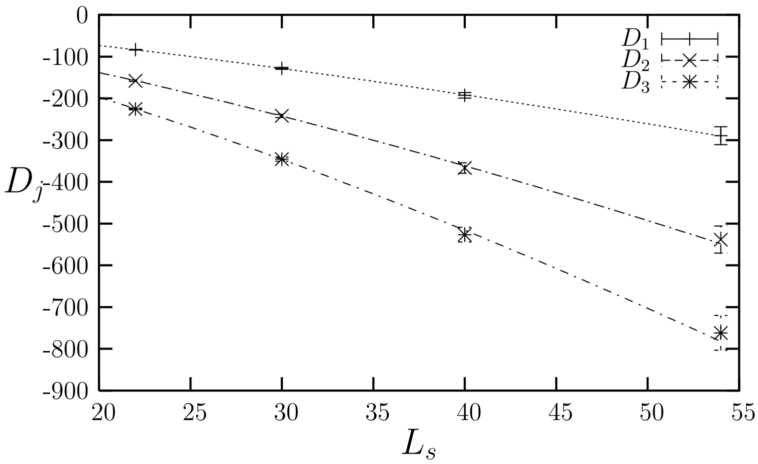

Figure 3: D_1, D_2 and D_3 vs. L_s at $\beta = 0.52589$.

fits, the value of ν decreases and the quality of the new fits deteriorates as shown in table 2. The outcome of the fits to $D_j \sim L_s^{1/\nu}$ at $\beta = 0.52589$ is shown in table 3 and plotted in figure 3. As expected, these results are consistent with what we extracted from the global fits. The results of the analysis of the logarithmic derivatives confirm the conclusion of the previous paragraph that large lattices are required in order to reach the asymptotic FSS regime.

In this paragraph we present the FSS analysis of the effective order parameter $\langle\Phi\rangle$. We fitted the $L_s \geq 22$ data in the vicinity of the transition to a scaling function obtained from the Taylor expansion of eq. (3.2), up to a linear term,

$$\langle\Phi\rangle \simeq [c_1 + c_2(\beta_c - \beta)L_s^{1/\nu}]L_s^{\beta_{\text{mag}}/\nu}\,. \tag{4.5}$$

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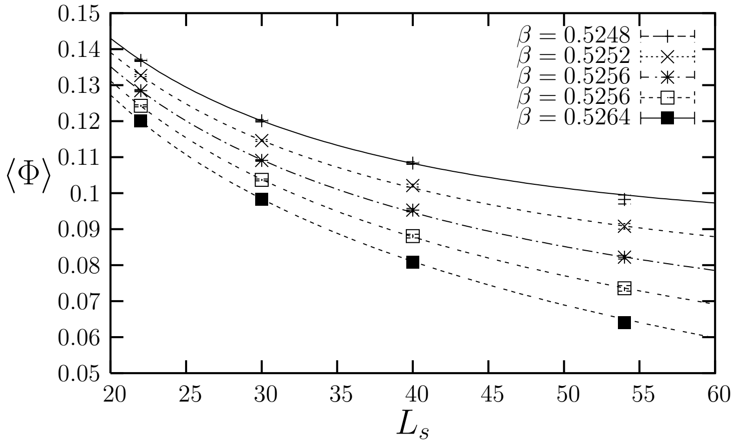

Figure 4: Effective order parameter $\langle \Phi \rangle$ vs. L_s at different values of β.

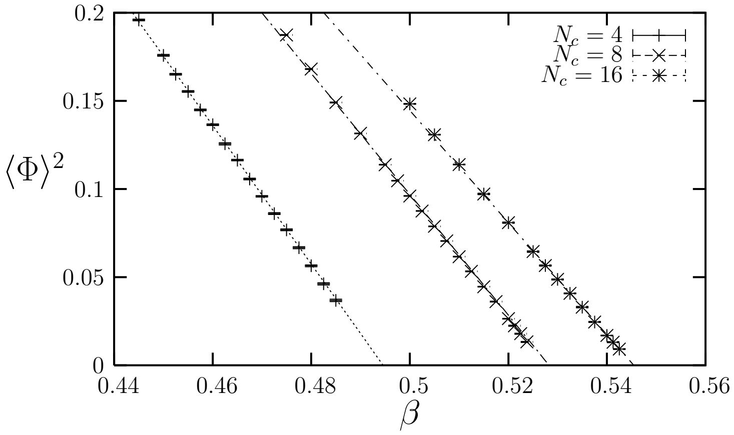

Figure 5: $\langle \Phi \rangle^2$ vs. β for $N_c = 4, 8$, and 16. The straight lines represent fits to the data in the mean field regions of the different N_c's.

We extracted $\nu = 0.740(16)$, $\beta_{\text{mag}}/\nu = 0.566(8)$, and $\beta_c = 0.52593(4)$ with $\chi^2/DOF = 0.43$. The measured value of ν is in good agreement with the $3d$ O(4) result, whereas the value of β_{mag}/ν deviates from the $3d$ O(4) result $\beta_{\text{mag}}/\nu = 0.5129(11)$ [19]. This discrepancy in the value of β_{mag} might be due to higher order corrections to the FSS relation. In figure 4 we plot data used in this global fit versus the lattice spatial size L_s at certain values of β. After including the quadratic term in the Taylor expansion of the FSS relation we got results consistent with those extracted from the linear expansion. Furthermore, attempts to add extra terms which could take into account corrections to FSS failed. The increase in the number of fitting parameters resulted in bad signals for the various parameters.

4.2 Broken phase

We performed simulations in the broken phase in order to study the dependence of $\langle \Phi \rangle$ on β at three different values of $N_c = 4, 8$, and 16. According to analytical and numerical evidence [11], the non-trivial scaling region of the finite temperature transition in the three-

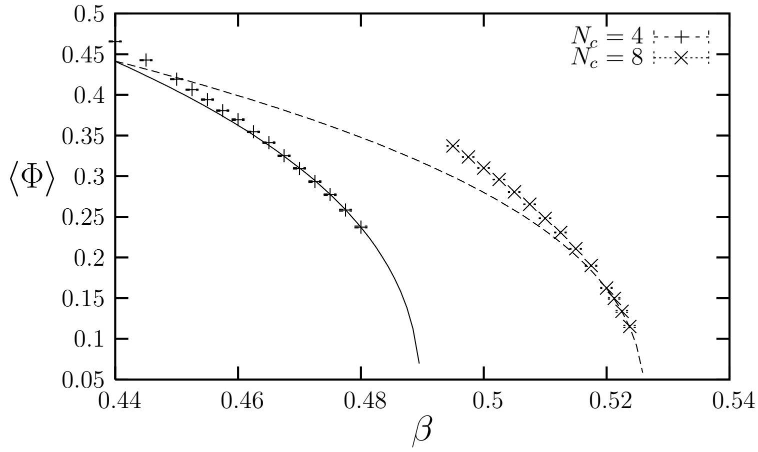

Figure 6: $\langle\Phi\rangle$ vs. β. The continuum and dashed lines represent fits to the data in the $3d$ O(4) scaling regions of $N_c = 4, 8$ repsectively.

dimensional Gross-Neveu model is suppressed by a factor of $1/\sqrt{N_c}$. The reasoning behind this result is that for finite N_c, close enough to T_c where the bosonic fluctuations become important, MF theory breaks down due to self-inconsistency. Hence, although the fermions do not affect the universal critical properties of the theory, they do influence the width of the actual critical region. It was also shown in [11] that in four-dimensions the non-trivial scaling region is suppressed by a factor of $1/[N_c \ln(L_t)]$. The logarithmic dependence on L_t is an outcome of the triviality of the four-dimensional theory.

We performed simulations on lattices with fixed $L_t = 4$ and $L_s = 54$. The Landau-Ginsburg MF theory predicts $\beta_{\text{mag}} = 1/2$. Therefore, in the MF regions of different N_s's the data for $\langle\Phi\rangle^2$ are expected to fit well to straight lines: $\langle\Phi\rangle^2 \sim (\text{const.} - \beta)^1$. The results for $\langle\Phi\rangle^2$ versus β are presented in figure 5. The $N_c = 16$ data shown in figure 5 fit well to the MF scaling for all the values of β. In the $N_c = 4$ and 8 cases however, for sufficiently small $\langle\Phi\rangle$, there is a crossover from MF scaling into the $3d$ O(4) universality. This is demonstrated more clearly in figure 6 where it is shown that the data for small values of $\langle\Phi\rangle$, fit well to

$$\langle\Phi\rangle = a(\beta_c - \beta)^{\beta_{\text{mag}}}, \tag{4.6}$$

where β_{mag} is fixed to its $3d$ O(4) value of 0.3836 [19]. The continuum and dashed lines in figure 6 represent fits to the data in the non-trivial scaling regions for $N_c = 4$ and $N_c = 8$, respectively. For $N_c = 4$, we extracted $\beta_c = 0.4901(3)$ by fitting the data in the range $0.470 \leq \beta \leq 0.480$. For $N_c = 8$, we extracted $\beta_c = 0.5263(1)$ by fitting the data for couplings $0.52000 \leq \beta \leq 0.52375$. In table 3 we show the χ^2/DOF for different sets of data fitted to eq. (4.6). It is clear that the fit quality deteriorates as we add data points at lower temperatures. Hence, we conclude that for $N_c = 4$ the crossover out of the $3d$ O(4) scaling region occurs roughly at $\beta_{\text{cross}} = 0.465$ with $\langle\Phi\rangle_{\text{cross}} = 0.34$ and for $N_c = 8$ at $\beta_{\text{cross}} = 0.5175$ with $\langle\Phi\rangle_{\text{cross}} = 0.190$. At this point we should note that in the MF region the order parameter is equal to the thermal fermion mass [8]. Therefore, it acts as an inverse correlation length. Given that our definition of the crossover

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No. of points	β range	χ^2/DOF
	$N_c = 4$	
4	0.4725-0.4800	0.10
5	0.4700-0.4800	0.17
6	0.4675-0.4800	0.58
7	0.4650-0.4800	2.50
8	0.4625-0.4800	3.2
	$N_c = 8$	
3	0.52125-0.52375	0.08
4	0.52000-0.52375	0.07
5	0.5175-0.52375	7.3

Table 4: Simulations for the FSS analysis.

	$3d$ O(4) spin model exponents	This study
ν	0.7479(90)	0.75(4)
γ	1.477(18)	1.46(6)
$\beta_{\rm mag}$	0.3836(46)	0.38(4)

Table 5: Critical exponents of the $3d$ O(4) Heisenberg model obtained in [19] and critical exponents obtained from this study.

from one scaling region to another is a bit loose, we believe that our results are consistent with the analytical prediction that the scaling region is suppressed by a factor of $1/N_c$ [11].

We also fitted the $N_c = 8$ effective order parameter data for β =0.5200, 0.52125, and 0.5225 to eq. (4.6) with the critical coupling fixed to $\beta_c = 0.52589$ (extracted from the FSS analysis of the susceptibility). We found $\beta_{\rm mag}$ to be 0.350(15), which is roughly two standard deviations below the $3d$ O(4) result $\beta_{\rm mag} = 0.3836(46)$. We believe that this small discrepancy is due to finite size effects which cause an increase in the value of $\langle \Phi \rangle$. Given that β_c is fixed, the fitting fuction is forced to become more abrupt, resulting in a smaller value for $\beta_{\rm mag}$.

5. Conclusions

We showed with relatively good precision that the NJL model with four lattice spacings in the temporal direction undergoes a second order phase transition. The critical indices were extracted by employing FSS analysis in the vicinity of T_c and analysis of data generated on large lattices in the low temperature chirally broken phase. Our results are consistent with the $3d$ O(4) classical spin model exponents and hence in accordance with the dimensional reduction scenario [2]. We summarize our results in table 5 and compare them with those of the classical $3d$ O(4) Heisenberg spin model [19]. The values and the error bars of the exponents presented in table 5 take into account slightly different predictions for each exponent measured by different methods in this study.

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We also verified numerically a previous analytical result [11] that the actual critical region is suppressed by a factor of $1/N_c$. Outside this region, where fluctuations are negligible, we have Landau-Ginsburg MF scaling predicted by the large-N_c calculations. This result should be contrasted with recent numerical results in strong coupling QCD, where the width of the actual critical region does not depend on N_c [20].

Furthermore, it is useful to comment on the phenomenological significance of the parameters used in this simulation. Within the framework of the Hartree approximation (for details see [7]), it can be easily shown that if one fixes F_π to its experimental value of 93MeV and the constituent quark mass to a physically reasonable value of 400 MeV then the inverse lattice spacing a^{-1} at the critical coupling $\beta_c = 0.5259$ (for $L_t = 4$) of the $N_c = 8$ nonzero temperature phase transition is of the order of 1 GeV.

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